Practice Quiz 2.2: A freeway accident

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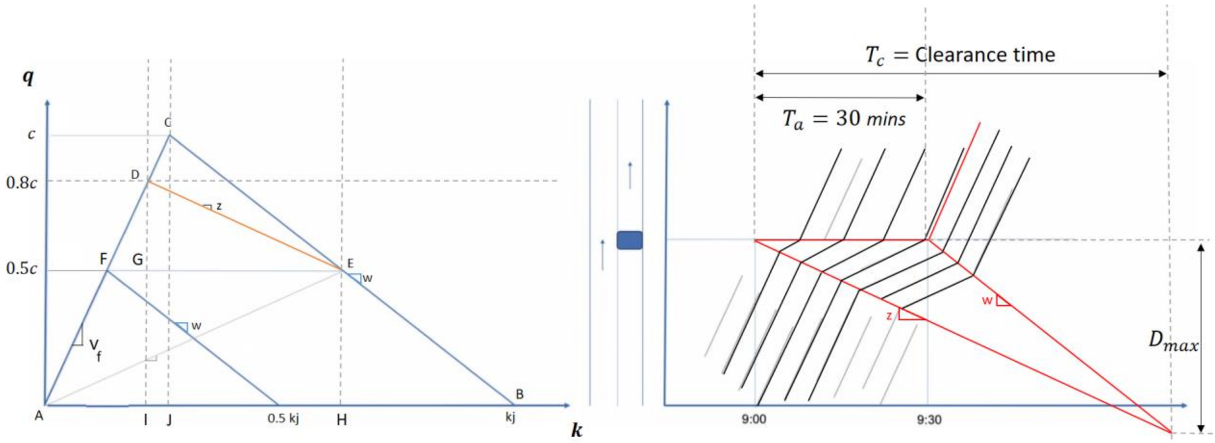
Solution

Firstly we must calculate the shockwave propagation speed, which is the slope of line DE on the Fundamental Diagram of the freeway shown below. When the accident occurs, one of the two lanes closes, so the maximum flow of the freeway decreases by 50%, i.e. the new $q'_{cr} = \frac{1}{2} q_{cr} = \frac{1}{2} c$ (point E). Similarly, as we assume a triangular FD, the jam density will also decrease by 50%, i.e. the new jam density is $k'_j = \frac{1}{2} k_j$. Since the FD is given, the following quantities are considered known: v_f, c, k_j, w (w can also be calculated based on the other three as will be shown).



In order to find the maximum distance that the queue propagates backwards, D_{max}, and the clearance time T_c we have to know the intersection point of the two red lines on the time space graph, which represent the creation and discharge of the shockwave. Based on this point, we can calculate the two quantities by using the time-space diagram and by applying simple principles of geometry:

The shockwave propagation speed is:

Slope of (DE): $z = \frac{DG}{GE} = \frac{0.8c - 0.5c}{IH}$

From triangle *ADI* : $v_f = \frac{DI}{AI} = \frac{0.8c}{AI} \Leftrightarrow AI = \frac{0.8c}{v_f}$ (1)

From triangle *EBH*: $w = \frac{EH}{HB} = \frac{0.5c}{HB} \Leftrightarrow HB = \frac{0.5c}{w}$ (2)

Also, we can see in the FD that:

$k_j = (AI) + (IH) + (HB) \Leftrightarrow (IH) = k_j - (AI) - (HB)$ (3)

By replacing Eq. (1) and (2) in (3) we have:

$(IH) = (GE) = k_j - \frac{0.8c}{v_f} - \frac{0.5c}{w}$ (4)

Based on Eq. (4) the slope of (DE) is:

$$z = \frac{0.3c}{k_j - \frac{0.8c}{v_f} - \frac{0.5c}{w}}$$

Also, for v_f and w we can write:

$$\left.\begin{aligned} v_f &= \frac{CJ}{AJ} = \frac{c}{AJ} \Leftrightarrow AJ = \frac{c}{v_f} \\ w &= \frac{CJ}{JB} = \frac{c}{k_j - AJ} \end{aligned}\right\} \qquad w = \frac{c}{k_j - \frac{c}{v_f}} \Leftrightarrow k_j = \frac{c}{v_f} + \frac{c}{w}$$

By replacing with the last relation, we get:

$$z = \frac{0.3c}{\frac{0.2c}{v_f} + \frac{0.5c}{w}} = \frac{0.3\, v_f w}{0.2w + 0.5v_f} = \frac{3\, v_f w}{2w + 5v_f}$$

T_c = Clearance time (in minutes after 9.00 am)

We should keep in mind that slopes w, z, v_f (speed units) are calculated based on the units of the FD diagram. When these slopes are transferred to the time-space diagram, their values might need to be properly adjusted to fit conform to the diagram's units, e.g. in this case (distance units)/min.

From the time-space diagram we can write:

$$w = \frac{D_{max}}{T_c - T_a} \Leftrightarrow D_{max} = w\,(T_c - T_a), \quad \text{where} \quad T_a = 30 \text{ mins}$$

$$z = \frac{D_{max}}{T_c} \Leftrightarrow D_{max} = z\, T_c$$

By replacing D_{max} in the first relationship we get:

$$z\, T_c = w\,(T_c - T_a) \Leftrightarrow T_c\,(w - z) = T_a w \Leftrightarrow T_c = \frac{T_a w}{w - z} = \frac{T_a w}{w - \frac{3v_f w}{2w + 5v_f}} = \frac{T_a w(2w + 5v_f)}{w\,(2w + 5v_f) - 3v_f w}$$

$$T_c = \frac{T_a(2w + 5v_f)}{2w + 2v_f} \Leftrightarrow T_c = T_a\left(1 + \frac{3}{2}\frac{v_f}{w + v_f}\right) = \mathbf{30 + 45}\frac{v_f}{w + v_f} \ \textbf{(mins)}$$

The maximum distance that the queue will expand upstream of the position of the accident before it fully disappears is:

$$D_{max} = w\,(T_c - T_a) = w\left(T_a + \frac{3}{2}\frac{T_a v_f}{w + v_f} - T_a\right) \Leftrightarrow \boldsymbol{D_{max}} = \frac{3}{2}\frac{T_a v_f w}{w + v_f} = \frac{45 v_f w}{w + v_f} \ \textbf{(dist units)}$$